GEOMETRIC NUMERICAL INTEGRATION

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The lectures of this summer school treat numerical methods that preserve geometric properties of the flow of a differential equation: symplectic integrators for Hamiltonian systems, symmetric integrators for reversible systems, methods preserving first integrals, etc. The main issue is long-time integration, which can be understood with the help of backward error analysis for ordinary differential equations, and with modulated Fourier expansions for problems with highly oscillatory solutions and for certain partial differential equations (such as nonlinearly perturbed wave equations).

The first two lectures treat:
– symplectic methods applied to Hamiltonian differential equations; symmetric integrators applied to reversible differential equations. In particular, symplectic Runge–Kutta methods, splitting methods, variational integrators will be discussed.
– an explanation of the long-time behaviour of numerical integrators using *backward error analysis*. This is done by considering a modified differential equation and a modified Hamiltonian (for symplectic methods). A recent application of backward error analysis provides insight into the near-preservation of energy for the Boris algorithm in charged particle dynamics.

The remaining two lectures will treat Hamiltonian systems with multiple time scales (highly oscillatory solutions):
– trigonometric time integrators for problems, where high oscillations originate from a linear part in the differential equations. Fermi–Pasta–Ulam-type problems are typical model problems. Besides the energy, which is exactly conserved, the oscillatory energy is nearly conserved (adiabatic invariant). Backward error analysis does not provide any information on the long-time behaviour, if the product of the time step size with the highest frequency in the system is not small.
– the main ingredient for a long-time analysis is the technique of *modulated Fourier expansion*. With a suitable ansatz, high oscillations are separated from the smooth motion in the system. This will be explained for the situation with a single high frequency, and then extended to (infinitely) many high frequencies (wave equations) and also to state-dependent high frequencies (charged particle dynamics with a strong magnetic field).

Most of the material for this summer school is taken from the monograph "Geometric Numerical Integration" (see below). The notes of a lecture on this topic, given at the Technische Universität München, are available at `http://www.unige.ch/~hairer/polycop.html`. The survey article "Dynamics, numerical analysis, and some geometry" (Christian Lubich's plenary talk at ICM 2018 in Rio de Janeiro) can be downloaded from `http://www.unige.ch/~hairer/preprints.html`.

Further references are:

E. Hairer, C. Lubich, G. Wanner, *Geometric Numerical Integration. Structure-Preserving Algorithms for Ordinary Differential Equations.* 2nd edition. Springer Series in Comput. Math., vol. 31, 2006.

J.M. Sanz-Serna, M.P. Calvo, *Numerical Hamiltonian Problems.* Chapman & Hall, Appl. Math. and Math. Comput., vol. 7, 1994.

B. Leimkuhler, S. Reich, *Simulating Hamiltonian Dynamics.* Cambridge Monographs on Applied and Computational Mathematics, vol. 14, 2004.

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