Mathematical modeling in biology.

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Leaky Integrate and Fire model

Leaky Integrate and Fire model :

- Neuron describe via its membrane potential $v \in (-\infty, V_F)$
- When the membrane potential reach the value V_F, the neuron spikes
- After a spike, the neuron, instantly, reset at the value V_R.

Model chosen (Brunel, Hakim) :

$$\frac{\partial p}{\partial t}(v,t) + \underbrace{\frac{\partial}{\partial v}\left[\left(-v + bN(t)\right)p(v,t)\right]}_{\text{Leaky Integrate and Fire}} - \underbrace{\sigma \frac{\partial^2 p}{\partial v^2}(v,t)}_{\text{noise}} = \underbrace{N(t)\delta(v - V_R)}_{\text{neurons reset}}, \qquad v \le V_F,$$

$$p(V_F,t) = 0, \quad p(-\infty,t) = 0, \quad p(v,0) = p^0(v) \ge 0 \quad N(t) := -\sigma \frac{\partial p}{\partial v}(V_F,t) \ge 0$$

- $p(v,t)$: density of neurons at time t with a membrane potential $v \in (-\infty, V_F)$
- b : strength of interconnexions.
- $N(t)$: Flux of neurons which discharge at time t.

Before studying this Equation, let us make some recall/study of simplest equations related to this one

The heat Equation on $\mathbb{R}$.

Heat equation: Let us consider the following Equation defined for $x \in \mathbb{R}$ by

$$\partial_t u(t,x) - \partial_{xx} u(t,x) = 0, \quad u(0) = u_0.$$

The solution can be written explicitly as

$$u(t,x) = \int_{-\infty}^{+\infty} K(t, x-y) u_0(y) dy$$

with

$$K(t,x) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

- K is a particular solution of the heat Equation
- We have

$$\lim_{t \to 0} K(t,x) = \delta_{x=0}.$$

The transport equation and advection equation ($d = 1$).

Let $V : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a smooth function.

Transport equation: The transport equation associated to V is given by

$$\partial_t u(t,x) + V(t,x)\partial_x u(t,x) = 0, \quad u(0,x) = u_0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+.$$

Advection equation: The advection equation associated to V is given by

$$\partial_t u(t,x) + \partial_x (V(t,x)u(t,x)) = 0, \quad u(0,x) = u_0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+.$$

The heat Equation on $(-\infty, V_F)$ with an external source as in the (NNLIF).

Let us consider the following Equation defined for $x \in (-\infty, V_F)$ by

$$\partial_t u(t,v) - \partial_{vv} u(t,v) = \delta_{v=V_R} N(t), \quad u(0) = u_0.$$

$$u(t, V_F) = 0, \quad -\partial_v u(t, V_F) = N(t).$$

We also have an explicit solution

$$u(t,x) = \int_{-\infty}^{V_F} K(t, x-y) u_0(y) dy + \int_0^t N(\tau) K(t-\tau, V_R - x) d\tau - \int_0^t N(\tau) K(t-\tau, V_F - x) d\tau.$$

Model chosen

$$\frac{\partial p}{\partial t}(v,t) + \underbrace{\frac{\partial}{\partial v}\left[\left(-v + bN(t)\right)p(v,t)\right]}_{\text{Leaky Integrate and Fire}} - \underbrace{\sigma \frac{\partial^2 p}{\partial v^2}(v,t)}_{\text{noise}} = \underbrace{N(t)\delta(v - V_R)}_{\text{neurons reset}}, \qquad v \leq V_F\,,$$

$$p(V_F,t) = 0, \qquad p(-\infty,t) = 0, \qquad p(v,0) = p^0(v) \geq 0\,.$$

$$N(t) := -\sigma \frac{\partial p}{\partial v}(V_F,t) \geq 0\,.$$

Questions :

- Qualitative dynamic and existence/uniqueness result (with Carrillo, Perthame, Smets, Caceres, Roux, Schneider) (see also Caceres, Carrillo, González, Gualdani, Perthame , Schonbek)
- Link between micro and macroscopic model (Delarue, Inglis, Rubenthaler, Tanré)
- Link with time elapsed model ? (Dumont, Henry, Tarniceriu)
- Add of heterogeneity (with B. Perthame and G. Wainrib)

Link with the time elapsed model in the linear case.

Link with the time elapsed model in the linear case with $K(s,u) = \delta_{s=0}$. (Dumont, Henry, Tarniceriu)

Term of discharge $d(s)$ in time elapsed : We compute d of Equation

$$\partial_t n + \partial_s n + d(s)n(s,t) = 0$$

corresponding to the one given by the Fokker-Planck equation.

Steps :

- We consider the function $q(s,v)$ solution of

$$\partial_s q(s,v) + \partial_v(-vq) - \sigma \partial_{vv} q = 0, \quad q(s=0,v) = \delta_{v=V_R}.$$

- d constructed via q using that the probability that a neuron reach the age s without discharge is

$$\mathcal{P}(a \geq s) = \int_{-\infty}^{V_F} q(s,v)dv = e^{-\int_0^s d(a)da}.$$

Link with the time elapsed model in the linear case.

Link kernel K : Density of probability $K(v,s)$ for a neuron to be at the potential v knowing that the time elapsed since its last discharge is $\geq s$,

$$K(v,s) := \frac{q(s,v)}{\int_{-\infty}^{V_F} q(s,v)dv}.$$

Formula of p with respect to n :

$$\text{If } \ p_0(v) := \int_0^{+\infty} K(v,s)n_0(s)ds, \ \text{ then } \ p(v,t) = \int_0^{+\infty} K(v,s)n(t,s)ds$$

is solution of

$$\partial_t p + \partial_v(-vp) - \sigma\partial_{vv} p = \delta_{v=V_R} N(t), \quad N(t) := -\sigma\frac{\partial p}{\partial v}(V_F,t), \quad p(0,v) = p_0.$$

with n solution of

$$\partial_t n + \partial_s n + d(s)n = 0, \quad n(0,s) = n_0(s).$$

Qualitative dynamic

$$\frac{\partial p}{\partial t}(v,t) + \underbrace{\frac{\partial}{\partial v}\left[\left(-v + bN(t)\right)p(v,t)\right]}_{\text{Leaky Integrate and Fire}} - \underbrace{a\frac{\partial^2 p}{\partial v^2}(v,t)}_{\text{noise}} = \underbrace{N(t)\delta(v - V_R)}_{\text{neurons reset}}, \qquad v \le V_F\,,$$

$$p(V_F,t) = 0, \qquad p(-\infty,t) = 0, \qquad p(v,0) = p^0(v) \ge 0\,.$$

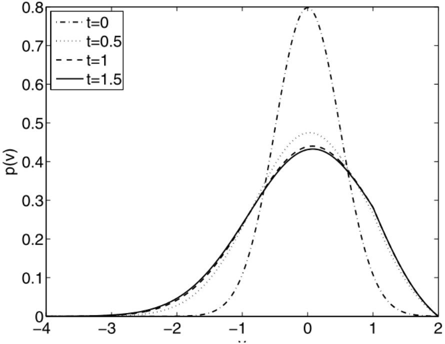
$$N(t) := -\sigma\frac{\partial p}{\partial v}(V_F,t) \ge 0\,.$$

Well posedness of the solution ?

The total activity of the network $N(t)$ acts instantly on the network.

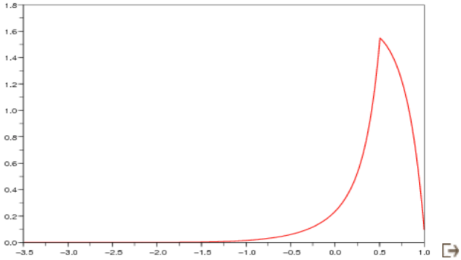
1. With the diffusion, this implies that for all $b > 0$, by well choosing the initial data, we have blow-up (Caceres, Carrillo, Perthame).
2. As soon $b \le 0$, the solution is globally well defined (Carrillo, González, Gualdani, Schonbek, Delarue, Inglis, Rubenthaler, Tanré).
3. If we add a delay N on the network, the equation is always well posed (with Caceres, Roux, Schneider)

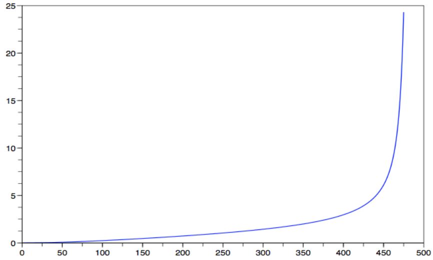
Qualitative dynamic



From Carrillo, Caceres, Perthame

Qualitative dynamic





Qualitative dynamic

Stationary states (Caceres, Carrillo, Perthame)

Implicit formula

$$p_\infty(v) = \frac{N_\infty}{a} e^{-\frac{(v-bN_\infty)^2}{2\sigma}} \int_{\max(v,V_R)}^{V_F} e^{\frac{(w-bN_\infty)^2}{2a}} dw$$

with the constraint on N_∞

$$\int_{-\infty}^{V_F} p_\infty(v)dv = 1.$$

1. There exists $C > 0$ such that, if $b \leq C$, there exists a unique stationary state
2. for intermediate b and some range of parameters (V_R, V_F, σ), there exists at least two stationary states
3. If b is big enough, there is no stationary states.

Qualitative dynamic

Asymptotic qualitative dynamic : if $b = 0$ (no interconnexions) solutions converge to a stationary state (Caceres, Carrillo, Perthame)

Idea of the proof :

- Entropy inequality with $G(x) = (x-1)^2$

$$\frac{d}{dt}\int_{-\infty}^{V_F} p_\infty(v) G\left(\frac{p(v,t)}{p_\infty(v)}\right) dv \leq -2\sigma \int_{-\infty}^{V_F} p_\infty(v) \left[\frac{\partial}{\partial v}\left(\frac{p(v,t)}{p_\infty(v)}\right)\right]^2 dv.$$

- Poincaré estimates

$$\int_{-\infty}^{V_F} \frac{(p-p_\infty)^2}{p_\infty} dv \leq C \int_{-\infty}^{V_F} p_\infty \left(\nabla\left(\frac{p-p_\infty}{p_\infty}\right)\right)^2 dv.$$

Entropy estimate

Classical entropy estimates : Let $G(x) = (x-1)^2$, then

$$\frac{d}{dt}\int_{-\infty}^{V_F} p_\infty(v) G\left(\frac{p(v,t)}{p_\infty(v)}\right) dv =$$

$$\underbrace{-N_\infty \left[G\left(\frac{N(t)}{N_\infty}\right) - G\left(\frac{p(V_R,t)}{p_\infty(V_R)}\right) - \left(\frac{N(t)}{N_\infty} - \frac{p(V_R,t)}{p_\infty(V_R)}\right) G'\left(\frac{p(V_R,t)}{p_\infty(V_R)}\right)\right]}_{\leq 0 \text{ because } G \text{ convex}}$$

$$-2\sigma \int_{-\infty}^{V_F} p_\infty(v) \left[\frac{\partial}{\partial v}\left(\frac{p(v,t)}{p_\infty(v)}\right)\right]^2 dv$$

$$\underbrace{+2b(N-N_\infty)\int_{-\infty}^{V_F} p_\infty \left[\partial_v\left(\frac{p(v,t)}{p_\infty(v)}\right)\left(\frac{p(v,t)}{p_\infty(v)} - 1\right) + \partial_v\left(\frac{p(v,t)}{p_\infty(v)}\right)\right] dv.}_{\text{non linear part}}$$

Qualitative dynamic

What happens if we add interconnexions ? (Carrillo, Perthame, Salort, Smets)

Inhibitory case :

- Inhibitory case : Uniform estimates on N in L^2, independent of b and the initial data,
- Inhibitory case : L^∞ estimates dependent of b and the initial data.

Exitatory case :

- Estimates on N, depending on the initial data and b.
- Convergence to a unique stationary state for sufficiently weak interconnections with respect to the initial data

Existence of periodic solutions ?

- Not numerically observed
- Signification of the blow-up condition ? Is there a way to prolongate the solution after the blow-up ?

A priori estimates on N.

Theorem :

Inhibitory case :

- There exists a constant C, such that for all initial data and $b \leq 0$, there exists $T > 0$ such that for all $I \subset [T, +\infty)$,

$$\int_I N(t)^2 dt \leq C(1 + |I|).$$

- Assume the initial data in L^∞. Then, for all $b \leq 0$, there exists $C > 0$ such that

$$\|N\|_{L^\infty} \leq C.$$

Excitatory case :

- Given an initial data and $b > 0$ small enough, $\exists$ $C > 0$ such that for all interval I,

$$\int_I N(t)^2 dt \leq C(1 + |I|)$$

Asymptotic dynamic.

Theorem :

Inhibitory case :

- Let $b \leq 0$. $\exists\ C, \mu > 0$ such that for all $0 \leq -b \leq C$ and all initial data

$$\int_{-\infty}^{V_F} p_\infty \left(\frac{p - p_\infty}{p_\infty}\right)^2 (t, v) dv \lesssim e^{-\mu t} \int_{-\infty}^{V_F} p_\infty \left(\frac{p - p_\infty}{p_\infty}\right)^2 (0, v) dv.$$

Excitatory case :

- Given an initial data, if $b > 0$ is small enough, then $\exists\ \mu > 0$ such that

$$\int_{-\infty}^{V_F} p_\infty \left(\frac{p - p_\infty}{p_\infty}\right)^2 (t, v) dv \lesssim e^{-\mu t} \int_{-\infty}^{V_F} p_\infty \left(\frac{p - p_\infty}{p_\infty}\right)^2 (0, v) dv.$$

Entropy estimate

Classical entropy estimates : Let $G(x) = (x-1)^2$, then

$$\frac{d}{dt}\int_{-\infty}^{V_F} p_\infty(v) G\left(\frac{p(v,t)}{p_\infty(v)}\right) dv =$$

$$\underbrace{-N_\infty \left[G\left(\frac{N(t)}{N_\infty}\right) - G\left(\frac{p(V_R,t)}{p_\infty(V_R)}\right) - \left(\frac{N(t)}{N_\infty} - \frac{p(V_R,t)}{p_\infty(V_R)}\right) G'\left(\frac{p(V_R,t)}{p_\infty(V_R)}\right)\right]}_{\leq 0 \text{ because } G \text{ convex}}$$

$$-2\sigma \int_{-\infty}^{V_F} p_\infty(v) \left[\frac{\partial}{\partial v}\left(\frac{p(v,t)}{p_\infty(v)}\right)\right]^2 dv$$

$$\underbrace{+2b(N - N_\infty)\int_{-\infty}^{V_F} p_\infty \left[\partial_v\left(\frac{p(v,t)}{p_\infty(v)}\right)\left(\frac{p(v,t)}{p_\infty(v)} - 1\right) + \partial_v\left(\frac{p(v,t)}{p_\infty(v)}\right)\right] dv.}_{\text{non linear part}}$$

Entropy estimates.

Strategy to obtain uniform estimates (inhibitory case)

Introduction of a fictif stationary state associated to a parameter $b_1 > 0$ different from $b \leq 0$.

For all convex function G regular,

$$\frac{d}{dt} p^1_\infty(v) G\left(\frac{p(v,t)}{p^1_\infty(v)}\right) =$$

$$-N^1_\infty \delta_{v=V_R}\left[G\left(\frac{N(t)}{N^1_\infty}\right) - G\left(\frac{p(v,t)}{p^1_\infty(v)}\right) - \left(\frac{N(t)}{N_\infty} - \frac{p(v,t)}{p^1_\infty(v)}\right) G'\left(\frac{p(v,t)}{p^1_\infty(v)}\right)\right]$$

$$-\sigma p^1_\infty(v)\, G''\left(\frac{p(v,t)}{p^1_\infty(v)}\right)\ \left[\frac{\partial}{\partial v}\left(\frac{p(v,t)}{p^1_\infty(v)}\right)\right]^2$$

$$+(bN(t) - b_1 N^1_\infty)\frac{\partial}{\partial v} p^1_\infty(v)\left[G\left(\frac{p(v,t)}{p^1_\infty(v)}\right) - \frac{p(v,t)}{p^1_\infty(v)} G'\left(\frac{p(v,t)}{p^1_\infty(v)}\right)\right].$$

Idea of proof for uniform estimates.

We choose $G(x) = x^2$, $b_1 > 0$ given, we multiply by a function γ supported on $(V_R, V_F]$, to have

$$\frac{d}{dt}\int_{-\infty}^{V_F} p_\infty^1 \left(\frac{p}{p_\infty^1}\right)^2 (t,v)\gamma(v)dv =$$

$$\int_{-\infty}^{V_F}(-v+bN(t))p_\infty^1\left(\frac{p}{p_\infty^1}\right)^2(t,v)\gamma'(v)dv - \frac{N^2(t)}{N_\infty^1}(t)\gamma(V_F)$$

$$-2\sigma\int_{-\infty}^{V_F} p_\infty^1\left(\partial_v\left(\frac{p}{p_\infty^1}\right)\right)^2\gamma(v)dv + \sigma\int_{-\infty}^{V_F} p_\infty^1\left(\frac{p}{p_\infty^1}\right)^2(t,v)\gamma''(v)dv$$

$$-\left(bN(t) - b_1 N_\infty^1\right)\int_{-\infty}^{V_F}\gamma(v)\partial_v p_\infty^1\left(\frac{p}{p_\infty^1}\right)^2 dv.$$

Sursolution methods.

We assume that $b \leq 0$ and that $0 \leq V_R < V_F$.

Definition

Let $b \leq 0$, $V_0 \in [-\infty, V_F)$ and $T > 0$. A function $\bar{p}$ is a universel sur-solution on $[V_0, V_F] \times [0, T]$ if

$$\frac{\partial \bar{p}}{\partial t}(v,t) - \frac{\partial}{\partial v}(v\,\bar{p}(v,t)) - a\frac{\partial^2 \bar{p}}{\partial v^2}(v,t) \geq \bar{N}(t)\delta(v - V_R) \quad (1)$$

on $(V_0, V_F) \times (0, T)$, where $\bar{N}(t) := -a\frac{\partial \bar{p}}{\partial v}(V_F, t) \geq 0$ and

$$\bar{p}(\cdot, t) \text{ is decreasing on } [V_0, V_F] \quad \forall t \in [0, T].$$

Lemma

Let $V_0 \in (-\infty, V_F)$ and $T > 0$. Let $\bar{p}$ be an universal sur-solution on $[V_0, V_F] \times [0, T]$, and assume that

$$\bar{p}(v,0) \geq p(v,0) \quad \forall v \in [V_0, V_F] \qquad \textit{and that} \qquad \bar{p}(V_0,t) \geq p(V_0,t) \quad \forall t \in [0, T].$$

Then, $\bar{p} \geq p$ on $[V_0, V_F] \times [0, T]$ and if $\bar{p}(\cdot,0) - p(\cdot,0)$ non idendically equal to 0, then $\bar{p} > p$ on $(V_0, V_F) \times (0, T]$.

Sur-solution method.

We construct two classes of universal sur-solution

- $$P(v,t) = \begin{cases} \exp(t) & \text{pour } v \leq V_R, \\ \exp(t)\,\frac{V_F - v}{V_F - V_R} & \text{pour } V_R \leq v \leq V_F. \end{cases} \quad (2)$$
- We consider Q_1 and Q_2 solutions of

$$-aQ_1' - vQ_1 = a \quad \text{on } (V_R, V_F), \qquad Q_1(V_F) = 0, \quad (3)$$
$$-aQ_2' - vQ_2 = 0 \quad \text{on } (0, V_R), \qquad Q_2(V_R) = Q_1(V_R), \quad (4)$$

We define Q on $[0, V_F]$ equal to Q_1 on $[V_R, V_F]$ and equal to Q_2 on $[0, V_R]$.

Sursolution Method.

Strategy

- Via a change of variable, we reduce our equation to the linear heat equation on a domain which depends on time and this outside the singularity at $v = V_R$.
- We use the 2 universal sur-solutions and the regularizing effect on the heat equation to prove that the solution is under the universal sur-solution βQ for β big enough, where Q is prolongated by $Q(0)$ on $(-\infty, 0)$

Sursolution Method.

Change of variable Let $t_0 \geq 0$ and $T \geq t_0$. We set

$$q(y,\tau) = e^{-(t-t_0)} p\left(e^{-(t-t_0)} y + \int_{t_0}^{t} bN(s) e^{-(t-s)} ds, t\right) \text{ et } \tau = \frac{1}{2} e^{2(t-t_0)}.$$

The function q is solution of the heat Equation

$$\partial_t q - a \partial_{yy} q = 0$$

on Ω_{t_0} which is the set of (y,τ) such that

$$\frac{1}{2} e^{-2t_0} \leq \tau \leq \frac{1}{2} e^{2(T-t_0)}, \; y \neq \sqrt{2\tau} V_R - \int_0^{\frac{1}{2}\ln(2\tau)} bN(s+t_0) e^s ds$$

$$\text{and } y < \sqrt{2\tau} V_F - \int_0^{\frac{1}{2}\ln(2\tau)} bN(s+t_0) e^s ds.$$

Sursolution Method.

We arg by a contradiction argument

- Assume that there exists $t_0 \geq 1$ such that for all β big enough (we can chose $v_0 \leq 0$)

$$p(v_0, t_0) = \beta Q(v_0)$$

- Using that, on $[0, t_0]$, Q is a sursolution, we know that N is bounded.
- We show that the cylinder $\Gamma_{v_0,r}$

$$[v_0 - r, v_0 + r, \frac{1}{2} - \frac{r^2}{a}, \frac{1}{2}] \subset \Omega_{t_0}$$

with

$$r \leq \frac{1}{2}\exp(-\frac{1}{2})V_R \qquad \text{et} \qquad \frac{r^2}{a} \leq \min\left(\frac{1}{2}(1-\exp(-1)), \frac{1}{2}\frac{V_R}{V_R - 2ba\beta}\right).$$

- We use the regularizing effect

$$|q(v_0, \frac{1}{2})| \leq Kar^{-3}\|q\|_{L^1(\Gamma_{v_0,r})}.$$

Conclusion of instantaneous LIF model

- Equation ill posed as soon $b > 0$ if the initial data is well chosen.
- If $b > 0$ is small enough and the initial data well chosen, exponential convergence to the unique stationary state.
- In the inhibitory case, uniform estimates on $N(t)$ and exponential convergence for $|b|$ small enough.
- Question of proof of convergence to the unique stationary state open, for the inhibitory case and $|b|$ large
- Question of periodic solution is totally open.

Equation with transmission delay

$$\frac{\partial p}{\partial t}(v,t) + \underbrace{\frac{\partial}{\partial v}\left[\left(-v + bN(t-d)\right)p(v,t)\right]}_{\text{Leaky Integrate and Fire}} - \underbrace{\sigma \frac{\partial^2 p}{\partial v^2}(v,t)}_{\text{noise}} = \underbrace{\frac{R(t)}{\tau}\delta(v - V_R)}_{\text{neurons reset}}, \qquad v \leq V_F\,,$$

$$R'(t) + \frac{R}{\tau} = N(t)$$

$$p(V_F,t) = 0, \qquad p(-\infty,t) = 0, \qquad p(v,0) = p^0(v) \geq 0\,.$$

$$N(t) := -\sigma \frac{\partial p}{\partial v}(V_F,t) \geq 0\,.$$

Principal properties (Caceres, Perthame)

- Still blow-up
- Existence of odd stationary states for all $b > 0$ and unique stationary state for $b \leq C$, $C > 0$ small enough
- Exponential convergence to a unique stationary without connectivity.

Equation with delay

$$\frac{\partial p}{\partial t}(v,t) + \underbrace{\frac{\partial}{\partial v}\left[\left(-v + bN(t-d)\right)p(v,t)\right]}_{\text{Leaky Integrate and Fire}} - \underbrace{\sigma\frac{\partial^2 p}{\partial v^2}(v,t)}_{\text{noise}} = \underbrace{N(t)\delta(v-V_R)}_{\text{neurons reset}}, \qquad v \leq V_F\,,$$

$$p(V_F,t) = 0, \qquad p(-\infty,t) = 0, \qquad p(v,0) = p^0(v) \geq 0\,.$$

$$N(t) := -\sigma\frac{\partial p}{\partial v}(V_F,t) \geq 0\,.$$

Principal properties (with Caceres, Roux et Schneider)

- No more blow-up
- Existence and uniqueness of a global classical solution
- Exponential convergence to a unique stationary state as soon $|b|$ small enough (with same assumption as in the case without delay).

Equation with delay

Idea of proof for global existence :

- Via a change of variable, we obtain the following implicit equation on the flux N.
- Via a fix point argument, we obtain local existence
- We construct a super solution to obtain uniform estimates and conclude to global existence

Equation with delay

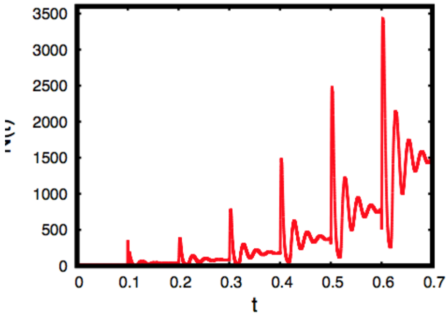
Construction of the supersolution for a given input N^0 :

$$\bar{\rho}(v,t) = e^{\xi t} f(v), \qquad \xi \text{ large enough}$$

Construction of f

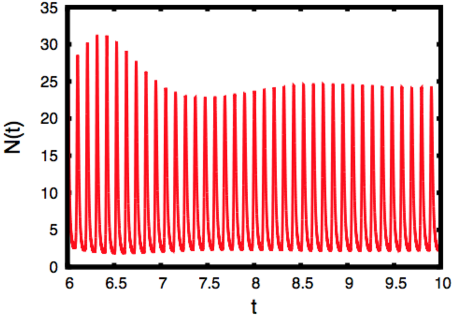
1. Let $\varepsilon > 0$ with $\frac{V_F+V_R}{2} + \varepsilon < V_F$ and let $\psi \in C_b^\infty(\mathbb{R})$ satisfying $0 \leq \psi \leq 1$ and
$$\psi \equiv 1 \text{ on } (-\infty, \frac{V_F+V_R}{2}) \text{ and } \psi \equiv 0 \text{ on } (\frac{V_F+V_R}{2} + \varepsilon, +\infty).$$
2. Let $B > 0$ such that
$$\forall t \geq 0, \forall v \in (V_R, V_F), \quad |-v + bN^0(t)| \leq B$$
and $\delta > 0$ such that $a\delta - B \geq 0$.
3. We chose
$$f \equiv 1 \text{ on } (-\infty, V_R]$$
$$f(v) = e^{V_R - v}\psi(v) + \frac{1}{\delta}(1-\psi(v))(1-e^{\delta(v-V_F)}) \text{ on } (V_R, V_F].$$

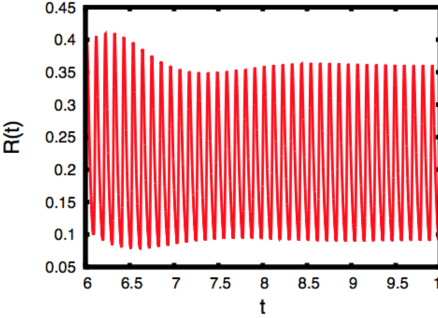
Equation with delay



from Caceres Schneider

Equation with delay





from Caceres Schneider